phenomena. For example, a gas moving under the action of partial pressure brings into motion a gas at rest which has no partial pressure. This is the case of a molecular ejector. The effectiveness of the performance of a molecular ejector can be determined using the theory expounded above. In a number of cases a gas can be set into motion which opposes its pressure gradient.

The theory developed here and the phenomena discovered play a major part in a number of practically important problems, and in particular in the problems of separating the gas and liquid mixtures by means of porous and semipermeable membranes.

REFERENCES

1. Struminskii, V. V., The effect of diffusion rate on the flow of gas mixtures. PMM Vol. 38, №2, 1974.
2. Hirschfelder, J. O., Curtiss, C. F. and Bird, R. B., Molecular Theory of Gases and Liquids. Izd. Inostr. Lit., Moscow, 1961.

Translated by L. K.

UDC 539.375

FRACTURE MECHANICS OF PIEZOELECTRIC MATERIALS. RECTILINEAR TUNNEL CRACK ON THE BOUNDARY WITH A CONDUCTOR

PMM Vol. 39, №1, 1975, pp. 149-159
B. A. KUDRIAVTSEV, V. Z. PARTON and V. I. RAKITIN
(Moscow)
(Received May 23, 1974)

A condition governing crack growth in a piezoelectric material is formulated, and the problem of tunnel crack development on the boundary between a piezoelectric ceramic and an elastic isotropic conductor is considered as an illustration. The stress components, displacements, electric field potential, displacement of the electric induction, and the magnitude of the critical load associated with crack growth are determined.

1. Fracture condition for piezoelectric media. The mechanical stress tensor components σ_{ij} in the static loading of a piezoelectric medium are functions of not only the geometric deformations but also of the electrical field.

Let us select the electrical field and the strain tensor components as independent variables, and let us represent the equation of the piezoelectric medium in crystal physics Cartesian x, y, z coordinates as follows [1]:

$$\sigma_{ij} = c_{ijkl}^E \varepsilon_{kl} - e_{ijk} E_k, \qquad D_i = e_{kli} \varepsilon_{kl} + \varepsilon_{ik}{}^s E_k \qquad (i, j, k, l = 1, 2, 3) \quad (1.1)$$

Here c_{ijkl}^E are the elastic moduli of the medium, e_{ijk} are the piezoelectric moduli, ε_{ik}^s are adiabatic dielectric constants of the medium, ε_{kl} are strain tensor components, σ_{ij} are stress tensor components, E_k are electrical field strength components, and D_i are the vector components of the electrical induction.

Neglecting volume forces and the Maxwell equations in the absence of free charges, the equilibrium equations of the medium are:

$$\partial\sigma_{ij}/\partial x_j = 0, \qquad \partial D_j/\partial x_j = 0 \tag{1.2}$$

To deduce the additional condition governing crack growth in a piezoelectric medium, let us introduce the external macroscopic energy flux $dA_{\Delta\Sigma}$ originating as a result of crack propagation.

Analogously to [2], we examine two possible states of a piezoelectric body (Fig. 1) corresponding to the instants t (solid lines) and $t_1 = t + \Delta t$ (dashed lines). The boundary conditions for the linearized problem can be formulated on the surface $\Sigma + \Delta\Sigma$ ($\Delta\Sigma$ is the increment of the bilateral surface of discontinuity). Let $\mathbf{u}$ $\{u_i\}$ denote the vector of displacement from a certain initial state to the state corresponding to the instant t, and $\mathbf{u}_1$ $\{u_{i1}\}$ the vector of the displacement from the same initial state to the state corresponding to the instant t_1.

On the basis of the equations of motion of the medium (1.2), the following equalities:

$$\frac{\partial}{\partial x_j}(\sigma_{ij1} + \sigma_{ij}) = 0, \qquad \frac{\partial}{\partial x_j}(D_{j1} - D_j) = 0$$

are satisfied for points of the volume V at the instants t and t_1 Multiplying the first of these equalities by $^1/_2\,(u_{i1} - u_i)$ and the second by $^1/_2(\varphi_1 + \varphi)$, we add the results and integrate over the whole body volume (φ is the electrical field potential, $\mathbf{E} = \mathrm{grad}(\varphi)$. After obvious manipulations, we obtain

$$\frac{1}{2}\int\limits_{\Sigma+\Delta\Sigma}(\sigma_{ij1} + \sigma_{ij})(u_{i1} - u_i)\,n_j dS + \frac{1}{2}\int\limits_{\Sigma+\Delta\Sigma}(D_{j1} - D_j)(\varphi_1 + \varphi)\,n_j dS = \tag{1.3}$$
$$\frac{1}{2}\int\limits_V [(\sigma_{ij1} + \sigma_{ij})(\varepsilon_{ij1} - \varepsilon_{ij}) + (D_{j1} - D_j)(E_{j1} + E_j)]\,d\tau$$

Using the relationship

$$\sigma_{ij}\varepsilon_{ij1} + E_j D_{j1} = \sigma_{ij1}\varepsilon_{ij} + E_{j1} D_j$$

which is verified directly by substitution of (1.1) therein, it is easy to show that the right side of (1.3) is the change in internal energy density

$$U = {}^1/_2\sigma_{ij}\varepsilon_{ij} + {}^1/_2 D_j E_j$$

when making the transition from one state to another. In the absence of an external heat flux, (1.3) can be written thus [2]:

$$dW = dA + dA_{\Delta\Sigma}$$

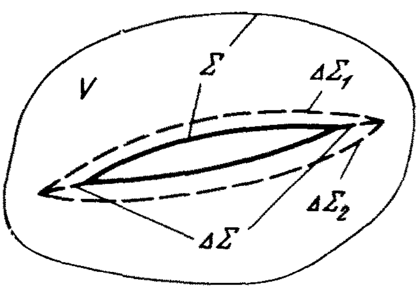

Fig. 1

Here dW is the increment in the internal energy, dA is the sum of the work of the external surface forces and the field on the whole boundary (up to the appearance of fracture) of the body Σ, and the expression for the energy flux in the formation of fracture $dA_{\Delta\Sigma}$ can be written as

$$dA_{\Delta\Sigma} = \frac{1}{2}\int\limits_{\Delta\Sigma}\sigma_{ij}u_{i1}n_j dS + \frac{1}{2}\int\limits_{\Delta\Sigma}\sigma_{ij}(u_{i1} - u_i)\,n_j dS + \tag{1.4}$$
$$\frac{1}{2}\int\limits_{\Delta\Sigma}\varphi D_{j1} n_j dS + \frac{1}{2}\int\limits_{\Delta\Sigma}\varphi_1 (D_{j1} - D_j)\,n_j dS$$

Taken into account in this latter expression is the fact that the following equalities

$$\int_{\Delta\Sigma} \sigma_{ij}u_in_jdS = 0, \qquad \int_{\Delta\Sigma} \varphi D_jn_jdS = 0$$

are valid because of the continuity of the displacements u_j, the stresses σ_{ij}, the potential φ and the electric induction D_j on $\Delta\Sigma$.

It is known from fracture mechanics [2 – 4] that the energy flux is related to the surface energy (γ is the intensity of the fracture surface energy) $dU_0 = \gamma\,(\Delta\Sigma_1 + \Delta\Sigma_2)$. The crack propagation condition is

$$dU_0 = - dA_{\Delta\Sigma} \tag{1.5}$$

in the statistical case and for an adiabatic process. In particular, if the crack edges are stress-free, condition (1.5) yields:

$$\gamma\,(\Delta\Sigma_1 + \Delta\Sigma_2) = -\frac{1}{2}\Big(\int_{\Delta\Sigma_1+\Delta\Sigma_2} \sigma_{ij}u_{i1}n_jdS + \int_{\Delta\Sigma_1+\Delta\Sigma_2} \varphi D_{j1}n_jdS + \tag{1.6}$$
$$\int_{\Delta\Sigma_1+\Delta\Sigma_2} \varphi_1 (D_{j1} - D_j)\, n_jdS$$

For example, the fracture condition (1.6) can be represented thus:

$$\gamma = -\frac{1}{2}\frac{d}{da}\Big(\int_0^a \sigma_{ij0}u_in_jdx + \int_0^a \varphi_0 D_jn_jdx + \int_0^a \varphi\,(D_j - D_{j0})\, n_jdx \tag{1.7}$$

for a crack located along the x-axis ($|\,x\,| \leqslant a$) in the plane strain case. Here σ_{ij0}, D_{j0}, φ_0 are parameters determined from the solution of the electroelasticity problem for the body region under consideration, but without a crack , and the integration is performed over both edges of the crack.

2. Tunnel crack on the boundary with a conductor. Formulation of the problem. Let us examine an unbounded half-space $z \geqslant 0$ of a piezoelectric material. A rectilinear crack is located in the $z = 0$ plane of isotropy of a transversely isotropic medium (texture of the class $\infty \cdot m$, crystals of hexagonal syngony of class $6 \cdot m$) on the boundary with an elastic isotropic conductor ($z \leqslant 0$), where the crack edges $-a \leqslant x \leqslant a$, $-\infty < y < \infty$ are load-free. A constant stress σ_0 parallel to the z-axis is given constant at infinity. The problem is considered for the plane strain case.

Following [1], it is possible to rewrite (1.1) in matrix form if the subscripts are replaced according to the following scheme:

$$11 \sim 1, \quad 23 = 32 \sim 4$$
$$22 \sim 2, \quad 13 = 31 \sim 5$$
$$33 \sim 3, \quad 12 = 21 \sim 6$$

The matrices of the elastic moduli c_{ij}^E, the piezoelectric moduli e_{ik}, and dielectric permeabilities ε_{kl}^s ($i, j = 1, 2 \ldots 6$; $k, l = 1, 2, 3$) for the textures and crystals under consideration are

$$
\|c_{ij}{}^E\| = \begin{Vmatrix} c_{11}{}^E & c_{12}{}^E & c_{13}{}^E & 0 & 0 & 0 \\ c_{12}{}^E & c_{11}{}^E & c_{13}{}^E & 0 & 0 & 0 \\ c_{13}{}^E & c_{13}{}^E & c_{33}{}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}{}^E & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44}{}^E & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}{}^E - c_{12}{}^E) \end{Vmatrix} \quad \|e_{ik}\| = \begin{Vmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{2.1}
$$

$$
\|\varepsilon_{kl}{}^s\| = \mathrm{diag}\,[\varepsilon_{11}{}^s, \varepsilon_{11}{}^s, \varepsilon_{33}{}^s]
$$

On the basis of (1.1), (2.1), the equations of a piezoelectric medium for the case of plane strain defined by the displacement $\mathbf{u}^+\ \{u^+(x, z), 0, w^+(x, z)\}$ $(z \geqslant 0)$ and the potential $\varphi(x, z)$ are

$$
\begin{aligned}
\overset{+}{\sigma}_{xx} &= c_{11}{}^E \frac{\partial u^+}{\partial x} + c_{13}{}^E \frac{\partial w^+}{\partial z} - e_{31}\frac{\partial \varphi}{\partial z} \\
\overset{+}{\sigma}_{zz} &= c_{13}{}^E \frac{\partial u^+}{\partial x} + c_{33}{}^E \frac{\partial w^+}{\partial z} - e_{33}\frac{\partial \varphi}{\partial z} \\
\overset{+}{\sigma}_{xz} &= c_{44}{}^E \left(\frac{\partial u^+}{\partial z} + \frac{\partial w^+}{\partial x}\right) - e_{15}\frac{\partial \varphi}{\partial x} \\
D_x &= e_{15}\left(\frac{\partial u^+}{\partial z} + \frac{\partial w^+}{\partial x}\right) + \varepsilon_{11}{}^s \frac{\partial \varphi}{\partial x} \\
D_z &= e_{31}\frac{\partial u^+}{\partial x} + e_{33}\frac{\partial w^+}{\partial z} + \varepsilon_{33}{}^s \frac{\partial \varphi}{\partial z}
\end{aligned} \tag{2.2}
$$

Taking account of (1.2), (2.2), we obtain the following fundamental equations to investigate the electroelasticity problem $(z \geqslant 0)$:

$$
\begin{aligned}
&c_{11}{}^E \frac{\partial^2 u^+}{\partial x^2} + c_{44}{}^E \frac{\partial^2 u^+}{\partial z^2} + (c_{13}{}^E + c_{44}{}^E)\frac{\partial^2 w^+}{\partial x \partial z} - (e_{31} + e_{15})\frac{\partial^2 \varphi}{\partial x \partial z} = 0 \\
&(c_{13}{}^E + c_{44}{}^E)\frac{\partial^2 u^+}{\partial x \partial z} + c_{44}{}^E \frac{\partial^2 w^+}{\partial x^2} + c_{33}{}^E \frac{\partial^2 w^+}{\partial z^2} - e_{15}\frac{\partial^2 \varphi}{\partial x^2} - e_{33}\frac{\partial^2 \varphi}{\partial z^2} = 0 \\
&(e_{31} + e_{15})\frac{\partial^2 u^+}{\partial x \partial z} + e_{15}\frac{\partial^2 w^+}{\partial x^2} + e_{33}\frac{\partial^2 w^+}{\partial z^2} + \varepsilon_{11}{}^s \frac{\partial^2 \varphi}{\partial x^2} + \varepsilon_{33}{}^s \frac{\partial^2 \varphi}{\partial z^2} = 0
\end{aligned} \tag{2.3}
$$

For the isotropic conducting medium $(z \leqslant 0)$ with the displacement $\mathbf{u}^-\ \{u^-(x, z), 0, w^-(x, z)\}$ for points of the medium, we have

$$
\begin{aligned}
\overset{-}{\sigma}_{xx} &= (\lambda + 2\mu)\frac{\partial u^-}{\partial x} + \lambda \frac{\partial w^-}{\partial z} \\
\overset{-}{\sigma}_{zz} &= (\lambda + 2\mu)\frac{\partial w^-}{\partial z} + \lambda \frac{\partial u^-}{\partial x}, \quad \overset{-}{\sigma}_{xz} = \mu\left(\frac{\partial w^-}{\partial x} + \frac{\partial u^-}{\partial z}\right)
\end{aligned} \tag{2.4}
$$

where λ, μ are Lamé coefficients, and the tensor components (2.4) satisfy the equilibrium equations (1.2).

By virtue of the linearity of (1.2), (2.2) – (2.4), the solution of the formulated static problem can be sought as the sum of solutions of the following two problems: the problem of determining the stress and strain states, the electric field components and the induction in a continuous piezoelectric medium reinforced everywhere in the plane with an isotropic medium subjected to the constant tensile stress σ_0 at infinity (A), and the problem of determining the states of media with a crack when external surface forces and a field act on the crack edges (B).

It is easy to verify that the solution of the problem A is

$$u^+ = u^- = \sigma_{xx}^+ = \sigma_{xz}^+ = \sigma_{xx}^- = \sigma_{xz}^- = E_x = D_x = 0 \qquad (2.5)$$

$$\sigma_{zz}^+ = \sigma_{zz}^- = \sigma_0, \qquad w^+ = \frac{e_{31}}{e_{31}c_{33}^E - e_{33}c_{13}^E}\sigma_0 z$$

$$w^- = \frac{1}{\lambda + 2\mu}\sigma_0 z, \qquad \varphi = \frac{c_{13}^E}{e_{31}c_{33}^E - e_{33}c_{13}^E}\sigma_0 z$$

$$D_z = \frac{e_{31}e_{33} + \varepsilon_{33}^s c_{13}^E}{e_{31}c_{33}^E - e_{33}c_{13}^E}\sigma_0$$

The solution of problem B can be obtained for the following conditions on the boundary between the two media:

$$\sigma_{zz}^+ = \sigma_{zz}^-, \quad \sigma_{xz}^+ = \sigma_{xz}^-, \quad \varphi = 0, \quad z = 0, \quad -\infty < x < \infty \qquad (2.6)$$

$$u^+ = u^-, \quad w^+ = w^-; \quad z = 0, \quad |x| > a$$

$$\sigma_{zz}^+ = -\sigma_0, \quad \sigma_{xz}^+ = 0; \quad z = 0, \quad |x| < a$$

$$u^+ = w^+ = \varphi = u^- = w^- = 0, \quad R = \sqrt{x^2 + z^2} \to \infty$$

3. System of singular integral equations. Let us seek the solution of (2.3) for $z \geqslant 0$ by using the Fourier integral transform

$$u^+(x, z) = \sqrt{\frac{2}{\pi}}\int_0^\infty U(p, pz)\sin px\, dp \qquad (3.1)$$

$$w^+(x, z) = \sqrt{\frac{2}{\pi}}\int_0^\infty W(p, pz)\cos px\, dp$$

$$\varphi(x, z) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(p, pz)\cos px\, dp, \quad z \geqslant 0, \quad x \geqslant 0$$

Substituting (3.1) into (2.3), we obtain a system of ordinary differential equations to determine the functions U, W, Φ. We write particular solutions of this system for $z \geqslant 0$ which satisfy the conditions at infinity as

$$U = \alpha e^{-kz}, \qquad W = \beta e^{-kz}, \qquad \Phi = \gamma e^{-kz}$$

Here k is the root with positive real part of the bicubic characteristic equation

$$\det \| a_{kl} \| = 0 \qquad (3.2)$$

$$a_{11} = c_{44}^E k^2 - c_{11}^E, \quad a_{12} = -a_{21} = (c_{13}^E + c_{44}^E)k, \quad a_{22} = c_{33}^E k^2 - c_{44}^E$$

$$a_{13} = -a_{31} = -(e_{31} + e_{15})k, \quad a_{23} = -a_{32} = -e_{33}k^2 + e_{15}, \quad a_{33} = \varepsilon_{33}^s k^2 - \varepsilon_{11}^s$$

An analysis of (3.2) shows that it has two real roots $\pm k_1$ and four pairwise conjugate complex roots $\pm \delta \pm i\omega$, for known piezoceramics in the media classes under consideration, where $k_1, \delta, \omega > 0$. The constants $\alpha(k)$, $\beta(k)$, $\gamma(k)$ which are a solution of the homogeneous system of equations with matrice $\| a_{kl} \|$ are defined by the formulas

$$\alpha = a_{12}a_{23} - a_{13}a_{22}, \quad \beta = -a_{11}a_{23} - a_{12}a_{13}, \quad \gamma = a_{11}a_{22} + a_{12}^2$$

Therefore, the general solution for U, W, Φ can be represented as

$$\begin{bmatrix} U \\ W \\ \Phi \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} A_1 e^{-k_1 z} + Re \left\{ \begin{bmatrix} \alpha_{21} + i\alpha_{22} \\ \beta_{21} + i\beta_{22} \\ \gamma_{21} + i\gamma_{22} \end{bmatrix} (B_1 + iC_1) e^{-z(\delta + i\omega)} \right\} \quad (3.3)$$

$$\alpha_1 = \alpha(k_1), \quad \beta_1 = \beta(k_1), \quad \gamma_1 = \gamma(k_1)$$
$$\alpha_{21} + i\alpha_{22} = \alpha(\delta + i\omega), \quad \beta_{21} + i\beta_{22} = \beta(\delta + i\omega)$$
$$\gamma_{21} + i\gamma_{22} = \gamma(\delta + i\omega)$$

Here $A_1(p)$, $B_1(p)$, $C_1(p)$ are functions to be determined from the boundary conditions.

Using (3.1) and (3.3), we obtain the following expressions for the displacement and potential:

$$u^+(x, z) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\alpha_1 A_1(p) e^{-k_1 pz} + (\alpha_{21} B_1(p) - \alpha_{22} C_1(p)) e^{-\delta pz} \times \quad (3.4)$$
$$\cos \omega pz + (\alpha_{22} B_1(p) + \alpha_{21} C_1(p)) e^{-\delta pz} \sin \omega pz] \sin px \, dp$$

$$w^+(x, z) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\beta_1 A_1(p) e^{-k_1 pz} + (\beta_{21} B_1(p) - \beta_{22} C_1(p)) e^{-\delta pz} \times$$
$$\cos \omega pz + (\beta_{22} B_1(p) + \beta_{21} C_1(p)) e^{-\delta pz} \sin \omega pz] \cos px \, dp$$

$$\varphi(x, z) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\gamma_1 A_1(p) e^{-k_1 pz} + (\gamma_{21} B_1(p) - \gamma_{22} C_1(p)) e^{-\delta pz} \times$$
$$\cos \omega pz + (\gamma_{22} B_1(p) + \gamma_{21} C_1(p)) e^{-\delta pz} \sin \omega pz] \cos px \, dp$$

On the basis of (2.2) and (3.4)

$$w^+(x, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\beta_1 A_1(p) + \beta_{21} B_1(p) - \beta_{22} C_1(p)] \cos px \, dp$$

$$u^+(x, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\alpha_1 A_1(p) + \alpha_{21} B_1(p) - \alpha_{22} C_1(p)] \sin px \, dp$$

$$\varphi(x, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\gamma_1 A_1(p) + \gamma_{21} B_1(p) - \gamma_{22} C_1(p)] \cos px \, dp$$

$$\sigma_{xz}^+(x, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty [m_1 A_1(p) + m_2 B_1(p) - m_3 C_1(p)] \, p \sin px \, dp$$

$$\sigma_{zz}^+(x, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \Big[\frac{m_1}{k_1} A_1(p) + \frac{m_2\delta + m_3\omega}{\delta^2 + \omega^2} B_1(p) -$$

$$\left.\frac{m_3\delta - m_2\omega}{\delta^2+\omega^2} C_1(p)\right] p \cos px \, dp$$

where we have introduced the notation

$$m_1 = e_{15}\gamma_1 - c_{44}{}^E (k_1\alpha_1 + \beta_1)$$
$$m_2 = e_{15}\gamma_{21} - c_{44}{}^E (\alpha_{21}\delta - \alpha_{22}\omega + \beta_{21})$$
$$m_3 = e_{15}\gamma_{22} - c_{22}{}^E (\alpha_{22}\delta + \alpha_{21}\omega + \beta_{22})$$

and used the equalities

$$c_{13}{}^E\alpha_1 - c_{33}{}^E k_1\beta_1 + e_{33}k_1\gamma_1 = \frac{m_1}{k_1}$$

$$c_{13}{}^E\alpha_{21} - c_{33}{}^E(\beta_{21}\delta - \beta_{22}\omega) + e_{33}(\gamma_{21}\delta - \gamma_{22}\omega) = \frac{m_2\delta + m_3\omega}{\delta^2+\omega^2}$$

$$c_{13}{}^E\alpha_{22} - c_{33}{}^E(\beta_{22}\delta + \beta_{21}\omega) + e_{33}(\gamma_{22}\delta + \gamma_{21}\omega) = \frac{m_3\delta - m_2\omega}{\delta^2+\omega^2}$$

We represent the solution of the equilibrium equations (1.2) for $z \leqslant 0$ as

$$u^-(x,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty [A_2(p) + B_2(p)pz] e^{pz} \sin px \, dp \tag{3.5}$$

$$w^-(x,z) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[-A_2(p) + B_2(p)\left(\frac{\lambda+3\mu}{\lambda+\mu} - pz\right)\right] e^{pz} \cos px \, dp$$

Here $A_2(P)$, $B_2(p)$ are functions to be determined from the boundary conditions (2.6).

On the basis of (2.4) we obtain by using (3.5)

$$u^-(x,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty A_2(p) \sin px \, dp \tag{3.6}$$

$$w^-(x,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[-A_2(p) + B_2(p)\frac{\lambda+3\mu}{\lambda+\mu}\right] \cos px \, dp$$

$$\sigma_{zz}{}^-(x,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty 2\mu\left[-A_2(p) + B_2(p)\frac{\lambda+2\mu}{\lambda+\mu}\right] p \cos px \, dp$$

$$\sigma_{xz}^-(x,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty 2\mu\left[A_2(p) - B_2(p)\frac{\mu}{\lambda+\mu}\right] p \sin px \, dp$$

Satisfying the continuity conditions (2.6) on the interface $z = 0$ of the two media, we obtain

$$C_1 = \frac{\gamma_1}{\gamma_{22}} A_1 + \frac{\gamma_{21}}{\gamma_{22}} B_1 \tag{3.7}$$

$$A_2 = \frac{\delta_1(\lambda+2\mu) + \delta_3\mu}{2\mu(\lambda+\mu)\gamma_{22}} A_1 + \frac{\delta_2(\lambda+2\mu)+\delta_4\mu}{2\mu(\lambda+\mu)\gamma_{22}} B_1$$

$$B_2 = \frac{\delta_1+\delta_3}{2\mu\gamma_{22}} A_1 + \frac{\delta_2+\delta_4}{2\mu\gamma_{22}} B_1$$

Here

$$\delta_1 = m_1\gamma_{22} - m_3\gamma_1, \quad \delta_2 = m_2\gamma_{22} - m_3\gamma_{21}$$

$$\delta_3 = \frac{m_1'}{k_1}\gamma_{22} - \frac{m_3\delta - m_2\omega}{\delta^2+\omega^2}\gamma_1, \quad \delta_4 = \frac{(m_2\delta + m_3\omega)\gamma_{22} - (m_3\delta - m_2\omega)\gamma_{21}}{\delta^2+\omega^2}$$

We introduce the functions

$$w(x) = w^+(x,0) - w^-(x,0), \quad u(x) = u^+(x,0) - u^-(x,0)$$

and by satisfying the remaining conditions in (2.6) we obtain a system of dual integral equations for the functions $A_1(p)$, $B_1(p)$ (3.8)

$$\sigma_{xz}^+(x,0) = \sqrt{\frac{2}{\pi}}\frac{1}{\gamma_{22}}\frac{d}{dx}\int_0^\infty [\delta_1 A_1(p) + \delta_2 A_2(p)]\cos px\,dp = 0, \quad 0 \leqslant x < a$$

(3.9)

$$\sigma_{zz}^+(x,0) = \sqrt{\frac{2}{\pi}}\frac{1}{\gamma_{22}}\frac{d}{dx}\int_0^\infty [\delta_3 A_1(p) + \delta_2 A_2(p)]\sin px\,dp = -\sigma_0, \quad 0 \leqslant x < a$$

$$w(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\gamma_{22}}\int_0^\infty [\delta_5 A_1(p) + \delta_6 B_1(p)]\cos px\,dp = 0, \quad x > a \qquad (3.10)$$

$$u(x) = \sqrt{\frac{2}{\pi}}\frac{1}{\gamma_{22}}\int_0^\infty [\delta_7 A_1(p) + \delta_8 B_1(p)]\sin px\,dp = 0, \quad x > a \qquad (3.11)$$

Here

$$\delta_5 = \beta_1\gamma_{22} - \beta_{22}\gamma_1 - \frac{\delta_1}{2(\lambda+\mu)} - \frac{\delta_3(\lambda+2\mu)}{2\mu(\lambda+\mu)}$$

$$\delta_6 = \beta_{21}\gamma_{22} - \beta_{22}\gamma_{21} - \frac{\delta_2}{2(\lambda+\mu)} - \frac{\delta_4(\lambda+2\mu)}{2\mu(\lambda+\mu)}$$

$$\delta_7 = \alpha_1\gamma_{22} - \alpha_{22}\gamma_1 - \frac{\delta_1(\lambda+2\mu)}{2\mu(\lambda+\mu)} - \frac{\delta_3}{2(\lambda+\mu)}$$

$$\delta_8 = \alpha_{21}\gamma_{22} - \alpha_{22}\gamma_{21} - \frac{\delta_2(\lambda+2\mu)}{2\mu(\lambda+\mu)} - \frac{\delta_4}{2(\lambda+\mu)}$$

Let us show that the system of dual integral equations (3.8) – (3.11) can be reduced to a system of singular integral equations with Cauchy kernels. We represent the relationships (3.10), (3.11) as

$$\int_0^\infty [\delta_5 A_1(p) + \delta_6 B_1(p)]\cos px\,dp = \begin{cases} \gamma_{22}\sqrt{\frac{\pi}{2}}\,w(x), & 0 \leqslant x < a \\ 0, & x > a \end{cases} \qquad (3.12)$$

$$\int_0^\infty [\delta_7 A_1(p) + \delta_8 B_1(p)]\sin px\,dp = \begin{cases} \gamma_{22}\sqrt{\frac{\pi}{2}}\,u(x), & 0 \leqslant x < a \\ 0, & x > a \end{cases}$$

and use the following formal representations of generalized functions:

$$\frac{2}{\pi}\int_0^\infty \sin pt \sin px\,dp = \delta(x-t), \qquad \frac{2}{\pi}\int_0^\infty \cos pt\cos px\,dp = \delta(x-t)$$

$$\frac{d}{dx}\left(\int_0^\infty \sin pt \sin px\frac{dp}{p}\right) = \frac{d}{dx}\left(\frac{1}{2}\ln\left|\frac{t+x}{t-x}\right|\right)$$

or

$$\int_0^\infty \sin pt \cos px \, dp = \frac{t}{t^2 - x^2}$$

We express $A_1(p)$ and $B_1(p)$ from (3.12) and convert (3.8), (3.9) to the form

$$g_{11} w(x) + g_{12} \frac{2}{\pi} \int_0^a \frac{u(t)\, t\, dt}{t^2 - x^2} = C_0 \tag{3.13}$$

$$-g_{21} \frac{2}{\pi} \int_0^a \frac{w(t)\, x\, dt}{t^2 - x^2} + g_{22} u(x) = \sigma_0 x, \quad 0 \leqslant x < a$$

(C_0 is a constant which will be determined below). Here

$$g_{11} = \frac{1}{\Delta}(\delta_7\delta_2 - \delta_8\delta_1), \qquad g_{12} = \frac{1}{\Delta}(\delta_6\delta_1 - \delta_5\delta_2)$$

$$g_{21} = \frac{1}{\Delta}(\delta_7\delta_4 - \delta_8\delta_3), \qquad g_{22} = \frac{1}{\Delta}(\delta_6\delta_3 - \delta_5\delta_4)$$

$$\Delta = \delta_5\delta_8 - \delta_6\delta_7$$

It is seen from (3.12) that $w(-x) = w(x)$ and $u(-x) = -u(x)$, and therefore

$$2\int_0^a \frac{w(t)\, x\, dt}{t^2 - x^2} = \int_{-a}^a \frac{w(t)\, dt}{t - x}, \qquad 2\int_0^a \frac{u(t)\, t\, dt}{t^2 - x^2} = \int_{-a}^a \frac{u(t)\, dt}{t - x}$$

Substituting these expressions into (3.13), we obtain the following system of singular integral equations with Cauchy kernels for the functions $w(x)$, $u(x)$:

$$g_{11} w(x) + g_{12} \frac{1}{\pi} \int_{-a}^a \frac{u(t)\, dt}{t - x} = C_0 \tag{3.14}$$

$$-g_{21} \frac{1}{\pi} \int_{-a}^a \frac{w(t)\, dt}{t - x} + g_{22} u(x) = \sigma_0 x \tag{3.15}$$

4. Solution of the electroelasticity problem. Let us turn to the solution of the system of integral equations (3.14), (3.15). Multiplying (3.15) by ig_1 and adding to (3.14), we obtain a single integral equation

$$f(x) + g \frac{1}{\pi i} \int_{-a}^a \frac{f(t)\, dt}{t - x} = \frac{C_0}{g_{11}} + \frac{ig_1}{g_{22}} \sigma_0 x \tag{4.1}$$

$$f(x) = w(x) + ig_1 u(x), \quad g_1 = \sqrt{\frac{g_{12} g_{22}}{g_{11} g_{21}}}, \quad g = \frac{g_{21}}{g_{22}} g_1$$

We note that for real piezoelectric and elastic media

$$g_{11} / g_{12} > 0, \quad g_{21} / g_{22} > 0, \quad g > 1$$

For example, for the piezoceramic composite PZT-4 [1] and steel (elastic modulus $E = 20 \times 10^{10}\, \mathrm{N/m^2}$ and Poisson's ratio $\nu = 0.25$)

$$g_{11} = 0.14 \times 10^{10}\, \mathrm{N/m^2}, \quad g_{12} = 6.1 \times 10^{10}\, \mathrm{N/m^2}, \quad g_{21} = 2.9 \times 10^{10}\, \mathrm{N/m^2}$$

$$g_{22} = 0.18 \times 10^{10}\, \mathrm{N/m^2}, \quad g_1 = 1.6, \quad g = 26$$

Following [5], to solve (4.1) we introduce the function

$$F(z)=\frac{1}{2\pi i}\int_{-a}^{a}\frac{f(t)\,dt}{t-z}$$

which is analytic in the complex plane with a crack along the segment $-a\leqslant x\leqslant a$ of the real axis. The boundary values of the continuous extension $F(z)$ on this segment to the left and right are determined by the Sokhotskii-Plemelj formulas

$$F^{+}(x)+F^{-}(x)=\frac{1}{\pi i}\int_{-a}^{a}\frac{f(t)\,dt}{t-x},\quad F(x)-F^{+}(x)=f(x) \tag{4.2}$$

After substituting (4.2) into (4.1), we obtain the Riemann boundary value problem

$$F^{+}(x)+\frac{g-1}{g+1}F^{-}(x)=\frac{1}{g+1}\left(\frac{C_0}{g_{11}}+i\frac{g_1}{g_{22}}\sigma_0 x\right) \tag{4.3}$$

Let us determine the particular solution of the homogeneous Riemann problem bounded near the ends $x=\pm a$ and vanishing thereon, as

$$X(z)=(z+a)^{1/2-i\varkappa}(z-a)^{1/2+i\varkappa},\qquad \varkappa=\frac{1}{2\pi}\ln\frac{g+1}{g-1}$$

Then the solution of the problem (4.3) bounded near the endpoints becomes

$$F(z)=\frac{X(z)}{2\pi i}\int_{-a}^{a}\frac{1}{g+1}\left(\frac{1}{g_{11}}C_0+i\frac{g_1}{g_{22}}\sigma_0 t\right)\frac{dt}{X^{+}(t)(t-z)} \tag{4.4}$$

Here $X^{+}(x)$ is the value of $X(z)$ on the left edge of the crack. Since the differences between the displacements w and u vanish at infinity in the problem under consideration, it should be required that $F(\infty)=0$, which results in the condition

$$\int_{-a}^{a}\left(\frac{1}{g_{11}}C_0+i\frac{g_1}{g_{22}}\sigma_0 t\right)\frac{dt}{X^{+}(t)}=0 \tag{4.5}$$

Using the methods of evaluating integrals [5], we obtain

$$C_0=\frac{g_1 g_{11}}{g_{22}}2\varkappa a\sigma_0 \tag{4.6}$$

and taking account of (4.6) we obtain the general solution of the boundary value problem (4.3) from (4.4) in the following form:

$$F(z)=-i\frac{g_1}{2gg_{22}}\sigma_0[X(z)-z+i2a\varkappa] \tag{4.7}$$

Substituting (4.7) into (4.2), we find

$$f(x)=w(x)+ig_1u(x)=-i\frac{g_1}{2gg_{22}}\sigma_0[X^{+}(x)-X^{-}(x)] \tag{4.8}$$

$$\int_{-a}^{a}\frac{f(t)\,dt}{t-x}=\frac{\pi g_1}{2gg_{22}}\sigma_0[X^{+}(x)+X^{-}(x)-2x+i4a\varkappa] \tag{4.9}$$

$$X^{\pm}(x)=\pm ie^{\mp\varkappa\pi}\sqrt{a^2-x^2}\left(\frac{a+x}{a-x}\right)^{\mp i\varkappa},\quad |x|<a$$

$$X^{+}(x)=X^{-}(x)=\sqrt{x^2-a^2}\left(\frac{x+a}{x-a}\right)^{-i\varkappa},\quad x>a$$

Therefore, the stresses, displacements, and electrical field components can be obtained explicitly at each point of the medium. In particular, the difference between the displacements of the crack edges $w(x)$ and the normal stresses $\sigma_{zz}(x, 0)$ are representable by the following expressions:

$$w(x) = \begin{cases} \frac{\sigma_0}{g_{21}} \operatorname{ch} \varkappa\pi \sqrt{a^2 - x^2} \cos\left(\varkappa \ln \frac{a-x}{a+x}\right), & |x| < a \\ 0, & x > a \end{cases} \quad (4.10)$$

$$\sigma_{zz}(x, 0) = \frac{d}{dx}\left(g_{21}\frac{1}{\pi}\int_{-a}^{a}\frac{w(t)\,dt}{t-x} - g_{22}u(x)\right) =$$

$$\begin{cases} -\sigma_0 & |x| < a \\ \frac{\sigma_0}{\sqrt{x^2-a^2}}\left[x\cos\left(\varkappa\ln\frac{x+a}{x-a}\right) + 2a\varkappa\sin\left(\varkappa\ln\frac{x+a}{x-a}\right)\right] - \sigma_0, & x > a \end{cases}$$

It follows from (4.10) that the displacement, stress and other physical quantities are oscillatory and change sign an infinity of times as x tends to the crack endpoints ($x = \pm a$).

For the piezoceramics presented in [1], the sections of sign-change are located in quite small neighborhoods of the crack endpoints $|x| \leqslant a$. The values of the parameter $d = (g+1)/(g-1)$ in (4.3) are less than three for a considerable number of piezoceramic composites with conductors (for example, $d = 1.08$ for the composite medium of the piezoceramic *PZT*-4 with steel, and $d = 1.03$ for copper). The estimate $|a - x| < 5 \times 10^{-4}a$ results [5] for neighborhoods in which the values of the physical quantities are oscillatory.

Therefore, the change in sign of the quantities under consideration occurs in that small neighborhood near the crack endpoints in which the solution obtained does not reflect the real state because of the departure from the linearized laws of a piezoelectric medium. We use condition (1.7) to determine the magnitude of the critical load.

Taking into account that

$$\int_{-a}^{a}\sqrt{a^2-x^2}\cos\left(\varkappa\ln\frac{a+x}{a-x}\right)dx = \frac{\pi a^2(1+4\varkappa^2)}{4\operatorname{ch}\varkappa\pi}$$

we obtain an expression relating the crack length to the applied load

$$\sigma_0 = \sqrt{\frac{8g_{21}\gamma}{\pi a(1+4\varkappa^2)}}$$

REFERENCES

1. Mason, W. P., ed., Physical Acoustics, Vol. 1, Pt. A. Academic Press, N. Y. - London, 1964.

2. Sedov, L. I., Mechanics of Continuous Media. "Nauka", Moscow, 1973. (See also Sedov, L. I., Mechanics of Continua (English translation), Pergamon Press, Book №09878, 1965.

3. Irwin, G. R., Analysis of stresses and strains near the end of a crack. J. Appl. Mech., Vol. 24, № 3, 1957.

4. Parton, V. Z. and Cherepanov, G. P., Fracture mechanics. In: 50 Years of Mechanics in the USSR, Vol. 3, "Nauka", Moscow, 1972.

5. Muskhelishvili, N. I., Some Basic Problems of the Mathematical Theory of Elasticity. Groningen, Noordhoff.

Translated by M. D. F.